Number Tricks

We started the first maths circle session with a look at a few number tricks. The puzzles are outlined on this page, with a hint or two for understanding. The explanation is on the next page.

- Take a three-digit number, e.g. 345, and write it out twice, to get 345,345.
  Now, take this number, and then divide it by 7. Divide this answer by 11. Finally, divide this answer by 13. What do you get?
  It turns out that you get:
    - $345345 \div 7 = 49335$
    - $49335 \div 11 = 4485$
    - $4485 \div 13 = 345$

  Amazingly, this seemed to work for everyone in the session, no matter what three-digit number they started off with! How can always happen?

  Hint: instead of going from 345345 to 345 by division, think about what you'd need to multiply 345 by to get 345345. Once you have an idea, check your hunch using the 7, 11, and 13.

  Once you've done this, see if you could make a similar trick where someone starts with a two digit number, or a four digit number, or a five digit number. Final question: why is the two-digit case not such a cool trick?

- Try multiplying out the following, and see if you can spot a pattern:
    - 1×1
    - 11×11
    - 111×111
    - 1111×1111

  It turns out there is a very distinctive pattern here, but can you explain why it works?

  Hint: It helps if you think of 11×11 as $(10 + 1) \times (10 + 1)$, and multiply out the brackets. Can you see the connection between this way of writing the multiplication, and long multiplication? In fact, for 111×111 and 1111×1111, it might be clearer to spot the pattern if you write out the long multiplication.

  At some point, the pattern breaks down – we made a guess in class as to when this would happen, but because the calculators didn't have enough room on the screen, we had to do the calculations on the computer. So, our old pattern stops – but could you figure out what the new one will be? And can you see if and when this pattern might break down?

- If today is Tuesday, what day will it be in 1,000 days' time? Remember, there are 7 days in a week…

  The key to this trick turns out to be remainders. For instance, it's the same idea as asking: if it's 3.30pm now, what time will it be in 1,000 hours' time? These kind of questions lead to the wonderful world of modulo arithmetic, which we will no doubt explore at some point soon…

Solutions:

- Let's go backwards, and ask: how would you get from 345 to 345,345?
  Well, $345{,}345 = 345{,}000 + 345$, which is 1,000 times 345, plus once 345.
  That is, $345{,}345 = 1{,}001 \times 345$.
  This means that $\frac{345{,}345}{1{,}001} = 345$.
  Finally, a quick check will tell us that $7 \times 11 \times 13 = 1{,}001$. This means that dividing by 7, then 11, then 13, is the same as dividing by 1,001.

  For the 2-digit number case, take e.g. 24 and 2,424, and ask: how do we get from 24 to 2,424?
  Well, $2{,}424 = 2{,}400 + 24$, i.e. one hundred time 24 plus 24.
  So $2{,}424 = 101 \times 24$, and the trick would ask people to divide by 101.
  The reason this mightn't be as impressive is that the 1,001 trick camouflages the 1,001 by using its factors – if you ask someone to divide by 101, it may give them an immediate hint as to how the trick works.

  For the 4-digit case you'd need to divide by 10,001. Luckily, $10{,}001 = 73 \times 137$, so you can disguise the 'magic number' somewhat.

  Similarly, for the 5-digit case we would use $100{,}001 = 11 \times 9{,}091$.

- You should notice the following pattern when you do the long multiplication:

$$\begin{array}{r} 1\,1\,1 \\ \underline{\times\,1\,1\,1} \\ 1\,1\,1 \\ 1\,1\,1\,0 \\ \underline{1\,1\,1\,0\,0} \\ 1\,2\,3\,2\,1 \end{array}$$

  You can see that there is one unit, two tens, three hundreds, two thousands, and one ten thousand. The same pattern is evident as you take bigger numbers. However, it breaks down after you've got 9 in the middle – so $1111111111 \times 1111111111$ won't follow this pattern.

  I'll leave I as an open question as to what the pattern turns into at this point.

- The idea is to divide by 7, and look at the remainder.
  So $1000 \div 7 = 142$, remainder 6. That means that 1,000 days is 142 full weeks, which brings us back to Tuesday, and then 6 more days, which brings us to Monday.

  The clock is a similar idea, but we divide by the 24 hours in a day:
  $1000 \div 24 = 41$, remainder 16. That means that 1,000 hours is 41 full (i.e. 24-hour) days, which brings us back to 3.30pm, and an extra 16 hours, which brings us on to 7.30am.